18.06 Problem Set 2

SOLUTIONS TO SELECTED PROBLEMS

1. Section 2.5, Problem 25

Answer: $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; the columns of B add up to 0, so B^{-1} does not exist.

2. Section 2.5, Problem 30

Answer: not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

3. Section 2.5, Problem 35

Answer: $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$; $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$; $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

4. Section 2.6, Problem 13

Answer: $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$. We need $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$.

5. Section 2.6, Problem 16

Answer: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} c = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \longrightarrow c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \longrightarrow$ $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

6. Section 2.6, Problem 19

Answer: $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU$;

$$\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = (\text{same } L) \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} (\text{same } U).$$

7. Section 2.6, Problem 28

Answer: LU will be impossible for $c = 6$ and $c = 7$ (a row exchange needed for $c = 6$).

8. Section 2.7, Problem 10

Answer: $(1, 2, 3, 4)$, $(2, 3, 1, 4)$, $(3, 1, 2, 4)$, $(2, 4, 3, 1)$, $(4, 1, 3, 2)$, $(3, 2, 4, 1)$, $(4, 2, 1, 3)$, $(1, 3, 4, 2)$, $(1, 4, 2, 3)$, $(2, 1, 4, 3)$, $(3, 4, 1, 2)$, $(4, 3, 2, 1)$.

9. Section 2.7, Problem 12

Solution: $(Px)^T(Py) = x^T P^T P y = x^T y$ because $P^T P = I$. In general $Px \cdot y = x \cdot P^T y \neq x \cdot Py$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

10. Section 2.7, Problem 16

Answer: $A^2 - B^2$ and ABA are symmetric if A and B are symmetric.

11. Section 2.7, Problem 19

Answer: (a) $(R^T AR)^T = R^T A^T (R^T)^T = R^T AR$, which is an n by n matrix.

(b) The j-th diagonal entry of $(R^T R)$ is equal to the dot product of the j-th row of R^T and the j-th column of R, i.e. the dot product of the j-th column of R with itself, which is positive.

12. Clearly, $A_\alpha A_\beta = A_{\alpha+\beta}$ since rotating a vector first by angle α and then by angle β is equivalent to rotating the vector by angle $\alpha + \beta$. The relation $A_\alpha A_\beta = A_{\alpha+\beta}$ shows that the set G of all A_α for $0 \leq \alpha < 2\pi$ is closed under multiplication. (Formally, we need to remark that if A_α and A_β are in G and $\alpha + \beta > 2\pi$, then $0 \leq \alpha + \beta - 2\pi < 2\pi$ and $A_\alpha A_\beta = A_{\alpha+\beta-2\pi}$, which is in G.) The set G contains the identity matrix $I = A_0$, and for every A_α in G, its inverse, namely $A_{2\pi-\alpha}$, is also in G (for $A_\alpha A_{2\pi-\alpha} = A_{2\pi-\alpha} A_\alpha = A_{2\pi} = I$). Thus G is a group of matrices.

13. Let $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. Set $AB = [c_{ij}]$ and $BA = [d_{ij}]$. First, suppose AB is symmetric. Then

$$c_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ik} a_{kj} = d_{ij}$$

for all i and j, hence $AB = BA$. On the other hand, if $AB = BA$, then

$$c_{ij} = d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = \sum_{k=1}^{n} a_{jk} b_{ki} = c_{ji}$$

for all i and j, hence AB is symmetric.