18.06 Problem Set 10

SOLUTIONS

1. Section 6.6, Problem 5

Answer: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by itself and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by itself.

2. Section 6.6, Problem 12

Solution: Suppose $M^{-1}JM = K$. Then

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}.$$

Equating the entries of the two matrices, we get in particular that $m_{21} = m_{22} = m_{23} = m_{24} = 0$, i.e. the entire second row of M is 0, so M is not invertible.

3. Section 6.6, Problem 20

Solution: (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$, so A^2 is similar to B^2.

(b) A may not be similar to $B = -A$, e.g. when A has eigenvalue λ and has no eigenvalue equal to $-\lambda$. In some cases, though, A is similar to $-A$, e.g. when all eigenvalues of A are equal to 0.

(c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because the eigenvalues are distinct.

(d) PAP^T is similar to A.

4. Section 6.6, Problem 21

Solution: We have

$$J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence J^2 has all eigenvalues equal to 0 and two independent eigenvectors, e.g. $v_1 = (1,0,0,0,0)$ and $v_2 = (0,1,0,0,0)$. Thus the Jordan form of J^2 contains two blocks with 0's on the diagonal. The block sizes are either 3 and 2, or else 4 and 1. Suppose J' is the Jordan form of J^2; then $(J')^3$ is similar to $(J^2)^3 = J^6 = 0$. If J' had block sizes 4 and 1, then $(J')^3$ would be nonzero and wouldn't be similar to J^4. Thus the correct Jordan form of J^2 has block sizes 3 and 2.

5. Section 6.7, Problem 7

Solution: $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $v_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Thus

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [u_1 \; u_2] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [v_1 \; v_2 \; v_3]^T.$$

6. Section 6.7, Problem 9

Answer: $A = 12UV^T$.

7. Section 6.7, Problem 10

Answer: $A = W\Sigma W^T$ is the same as $A = U\Sigma V^T$.

8. Section 6.7, Problem 13

Solution: Suppose the SVD of R is $R = U\Sigma V^T$. Then multiply by Q. So the SVD of this A is $(QU)\Sigma V^T$.

9. Section 6.7, Problem 14

Answer: The smallest change in A is to set the smallest singular value σ_2 to 0.

10. Section 6.7, Problem 16

Solution: The singular values of $A + I$ are not $\sigma_j + 1$. They come from the eigenvalues of $(A+I)^T(A+I)$.