18.06 Problem Set 1

SOLUTIONS

1. Section 2.1, Problem 10

Answer: $Ax = (18, 5, 0)$; $Ax = (3, 4, 5, 5)$.

2. (a) What two vectors are obtained by rotating the plane vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ by 30 degrees in the clockwise direction? Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30 degrees. (Problem 22 in Section 2.1 is helpful.)

(b) Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x + y + z = 0$. (Hint: try $v = (1, 0, 0)$ first.)

Solution. (a) Rotating $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ by 30 degrees in the clockwise direction yields vectors $e_1' = \begin{bmatrix}\sqrt{3}/2\\-1/2\end{bmatrix}$ and $e_2' = \begin{bmatrix}1/2\\\sqrt{3}/2\end{bmatrix}$. The desired matrix A has vectors e_1' and e_2' as columns, i.e.

$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

(b) The unit normal vector to the plane $x+y+z = 0$ is $\hat{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. To obtain the reflection of a vector v through this plane one needs to subtract from v twice the projection of v onto $\hat{u}$. The projection is given by $(v \cdot \hat{u})\hat{u}$, so the desired matrix B satisfies

$$Bv = v - 2(v \cdot \hat{u})\hat{u}.$$

Substituting the three basis vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ for v, we get

$$B\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{bmatrix}1-\frac{2}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{bmatrix};$$

$$B\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{bmatrix}-\frac{2}{3}\\1-\frac{2}{3}\\-\frac{2}{3}\end{bmatrix};$$

$$B\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\1-\frac{2}{3}\end{bmatrix}.$$

Hence the answer is

$$B = \begin{bmatrix}1/3 & -2/3 & -2/3\\-2/3 & 1/3 & -2/3\\-2/3 & -2/3 & 1/3\end{bmatrix}.$$

3. Section 2.2, Problem 21

Answer: Pivots: 2, 3/2, 4/3, 5/4. Solution: $t = 4$, $z = -3$, $y = 2$, $x = -1$.

4. Section 2.2, Problem 7

Answer: Elimination fails for $a = 2$ (no solution). A row exchange is necessary for $a = 0$. After the exchange, the solution is $x = 3$, $y = -1$.

5. Section 2.2, Problem 27

Answer: $s = 10$.

6. Section 2.3, Problem 17

Answer: The equations are

$$a + b + c = 4;$$

$$a + 2b + 4c = 8;$$

$$a + 3b + 9c = 14.$$

The solution to this system is $(a, b, c) = (2, 1, 1)$.

7. Section 2.3, Problem 18

Answer:

$$EF = \begin{bmatrix}1 & 0 & 0\\a & 1 & 0\\b & c & 1\end{bmatrix},\ FE = \begin{bmatrix}1 & 0 & 0\\a & 1 & 0\\b+ac & c & 1\end{bmatrix},\ E^2 = \begin{bmatrix}1 & 0 & 0\\2a & 1 & 0\\2b & 0 & 1\end{bmatrix},\ F^3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 3c & 1\end{bmatrix}.$$

8. Section 2.3, Problem 25

Answer: The first two rows of A add up to the bottom row, so in order for $Ax = b$ to have a solution, b must have the same property. Thus the last entry of b must be changed from 6 to 3.

9. Section 2.4, Problem 14

Answer: (a) True.
(b) False. If A is m by n, then B must be n by m, but m and n can be distinct.
(c) True.
(d) False. For example, if B is the matrix of all zeros, then A can be any matrix with appropriate dimensions.

10. Section 2.4, Problem 22

Answer: $A = A^2 = A^3 = \ldots$; $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$; $(AB)^2 = (AB)^3 = \ldots = 0$.

11. Section 2.4, Problem 22

Answer: (a) E.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
(b) E.g. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

12. Section 2.5, Problem 32

Answer: $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

13. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

Solution. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then

$$AB - BA = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} - \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix} =$$

$$= \begin{bmatrix} bg - fc & af + bh - eb - fd \\ ce + dg - ga - hc & cf - gb \end{bmatrix}.$$

Note that the sum of the diagonal entries of $AB - BA$ is 0, and the sum of the diagonal entries of I is 2, hence $AB - BA \neq I$ for any 2 by 2 matrices A and B.

Comment. The sum of the diagonal entries of a square matrix A is called the *trace* of A, denoted $\mathrm{tr}(A)$. For any two n by n matrices A and B, the equation $\mathrm{tr}(AB - BA) = 0$ holds; this can be checked in the same straightforward way as done above for $n = 2$. Since $\mathrm{tr}(I) = n$, we cannot have $AB - BA = I$ for any n by n matrices A and B.