18.06 Fall 2004 Quiz 2 November 15, 2004

Your name is:

Please circle your recitation:

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4. T10 K. Cheung
5. T10 Y. Rubinstein
6. T11 K. Cheung
7. T11 V. Angeltveit
8. T12 V. Angeltveit
9. T12 F. Rochon
10. T1 L. Williams
11. T1 K. Cheung
12. T2 T. Gerhardt

Grading:

Question	Points	Maximum
Name + rec		5
1		22
2		20
3		25
4		28
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation section correctly is worth 5 points.
The exam is worth a total of 100 points.

1. (22 pts.)

(a) For the following 3×3 matrix A, compute its determinant by using the cofactor formula and expanding along the third column. Show the values of the 3 cofactors you compute.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -2 \\ 1 & -4 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(A) &= 3(\det\left(\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\right)) - 2(-\det\left(\begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}\right)) + 1(\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right)) \\ &= 3(4-2) - 2(-(-4-2)) + 1(2-(-2)) \\ &= 3(2) - 2(6) + 1(4) \\ &= -2. \end{aligned}$$

The cofactors are of course the terms in the brackets in the penultimate line.

(b) Consider the matrix B obtained from A by adding rows 1 and 3 to row 2. Should $\det(B)$ equal $\det(A)$? Why?

Solution: Yes. Adding a multiple of one row to another does not change the determinant.

Compute $\det(B)$ directly from B.
Solution:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & -4 & 1 \end{bmatrix}$$

We'll expand along the second row because it looks easiest.

$$\begin{aligned} \det(B) &= 1(-(2+12)) + 0(\text{whatever}) + 2(-(-4-2)) \\ &= -14 + 12 \\ &= -2 \end{aligned}$$

Sure enough, this is the same as the answer for part (a).

2. (20 pts.) Give all values of x for which A has an eigenvalue equal to 2.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & x & 2 \\ x & -2 & 3 \end{bmatrix}.$$

Solution: The matrix A will have 2 as an eigenvalue if and only $\det(A - 2I) = 0$. Therefore, to achieve this, it is necessary and sufficient that x satisfy

$$\begin{aligned} 0 &= \det(A - 2I) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x & -2 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x+1 & 0 & 0 \end{bmatrix}\right) \qquad \text{(after adding row 1 to row 3)} \\ &= (x+1)(4 + x - 2) \\ &= (x+1)(x+2). \end{aligned}$$

Hence, A has 2 as an eigenvalue if and only if x equals -1 or -2.

3. (25 pts.)

(a) Use the Gram-Schmidt procedure to find an **orthonormal** basis of $C(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution:

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = a_2 - \frac{a_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} u_3 &= a_3 - \frac{a_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{a_3 \cdot u_2}{u_2 \cdot u_2} u_2 \\ &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now

$$q_1 = \frac{1}{\|u_1\|} u_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad q_2 = \frac{1}{\|u_2\|} u_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad q_3 = \frac{1}{\|u_3\|} u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}.$$

So an orthonormal basis for $C(A)$ is

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) Find the projection matrix P for projecting onto $C(A)$.

Solution: Recall that with the factorization $A = QR$, the projection matrix $P = A(A^TA)^{-1}A^T$ becomes QQ^T. Hence, we have

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(c) Check your answer for P by computing Pa_1 where a_1 is the first column of A.

Solution: We have

$$Pa_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a_1$$

as expected.

4. (28 pts.) For each of the following statements, determine if it is *always* true. If so, answer yes. Otherwise, answer no. Just circle yes or no. You do not need to justify your answer. However, we will be giving 4 points for a correct answer, 0 points for no answer at all, and **-2 points for an incorrect answer.**

   (a) **Yes — No.** Let F be the vector space of all 3×3 matrices. Let T be the transformation that maps $A \in F$ to $\det(A) + \text{trace}(A)$. Then T is a linear transformation.

   Solution: No (Reason: $T(2I) \neq 2T(I)$, for example)

   (b) **Yes — No.** Let A be the $m \times n$ (edge-node) incidence matrix of a connected graph with n vertices and m edges. Then the left nullspace of A has dimension $m - n + 1$.

   Solution: Yes (Reason: the rank of A is $n - 1$.)

   (c) **Yes — No.** An orthogonal matrix can have an eigenvalue equal to 0.

   Solution: No (Reason: an orthogonal matrix is non-singular as the columns are perpendicular and so certainly independent.)

   (d) **Yes — No** Let $\hat{x}$ be the least-squares solution to $Ax = b$. Then $b - A\hat{x}$ is orthogonal to the column space of A.

   Solution: Yes (By definition!)

   (e) **Yes — No.** Let P be the projection matrix for projecting onto a subspace F. Then $I - P$ is the projection matrix for projecting onto $F^{\perp}$.

   Solution: Yes (Reason: the sum of the projections must be the identity)

   (f) **Yes — No.** There are values for a, b, c, d, e, f such that the following matrix A satisfies $A^2 = 2A$:

$$A = \begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}.$$

   Solution: No (Reason: multiplying by any eigenvector, and then considering a non-zero entry of it and dividing out, we get that *any* eigenvalue must satisfy $\lambda^2 = 2\lambda$, ie, $\lambda = 0$ or 2. But the sum of the eigenvalues of A is $\text{trace}(A) = 3$ which cannot be obtained by adding 0's and 2's.)

   (g) **Yes — No.** Let 0 denote the 5×5 matrix whose entries are all 0. If $A^{10} = 0$ then $A = 0$.

   Solution: No (Counterexample: $A = 0$ except for $A_{1,10} = 1$.)