18.06, Fall 2004, Problem Set 9 Solutions

1. (5 pts) P is symmetric since $P = A(A^TA)^{-1}A^T$ (where we have kept only linearly independent columns of A if A did not have full column rank to start with). In lecture (and in the book), we saw that symmetric matrices are diagonalizable.

   We can also get explicitly a diagonalization. For this, take an orthonormal basis of $C(A)$, say $q_1, \cdots, q_r$ and also an orthonormal basis of $C(A)^\perp = N(A^T)$, say $q_{r+1}, \cdots, q_m$ and construct the $m \times m$ matrix Q whose columns are the q_i's. Q is orthogonal: $Q^T = Q^{-1}$. The diagonalization is now $A = Q\Lambda Q^{-1}$ where Λ is a diagonal matrix with $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 1$ and $\lambda_{r+1} = \cdots = \lambda_m = 0$.

2. (22 pts.) Consider the matrix:

$$A = \begin{bmatrix} 0.5 & b & 0 & a \\ a & 0.5 & b & 0 \\ 0 & a & 0.5 & b \\ b & 0 & a & 0.5 \end{bmatrix}$$

   (a) $a, b \geq 0$, $a + b = 0.5$

   (b)

$$\begin{bmatrix} 0.5\ b\ 0\ a \\ a\ 0.5\ b\ 0 \\ 0\ a\ 0.5\ b \\ b\ 0\ a\ 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5+b+a \\ 0.5+b+a \\ 0.5+b+a \\ 0.5+b+a \end{bmatrix} = (0.5+b+a) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

   so $0.5 + a + b$ is an eigenvalue. When $a + b = 0.5$, this is what we expected since every Markov matrix has an eigenvalue equal to 1.

   (c) $\omega = \pm i, \pm 1$.

   (d)

$$\begin{bmatrix} 0.5\ b\ 0\ a \\ a\ 0.5\ b\ 0 \\ 0\ a\ 0.5\ b \\ b\ 0\ a\ 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{bmatrix} = (0.5 + b\omega + a\omega^3) \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{bmatrix}.$$

   The corresponding eigenvalue is $0.5 + b\omega + a\omega^3$. Here are the 4 eigenvalues:

ω	eigenvalue
1	$0.5 + b + a$
-1	$0.5 - b - a$
i	$0.5 + i(b - a)$
$-i$	$0.5 - i(b - a)$

   (e) We just found 4 lineearly independent eigenvectors, so this means that for every eigenvalue its geometric multiplicity (given by the dimension of the nullspace of $A - \lambda I$) must be equal to its algebraic multiplicity. The eigenvalues are dictinct (and thus their geometric multiplicity is 1) for all values of a and b, except when $a + b = 0$ or when $a - b = 0$ for which $\lambda = 0.5$ is an eigenvalue of geometric multiplicity 2 (or 4 if $a = b = 0$). Thus, we have an eigenvalue (equal to 0.5) of algebraic multiplicity greater than 1 when $a = \pm b$.

(f) The determinant is the product of the eigenvalues. $\det A = (0.5 + a + b)(0.5 - (a + b))(0.5 + i(b - a))(0.5 - i(b - a)) = (0.25 - (a + b)^2)(0.25 + (a - b)^2)$.

(g) Yes, because A has four linearly independent eigenvectors for any values of a and b, and this is sufficient to guarantee that it is diagonalizable.

(h) The modulus of all eigenvalues of A should be less or equal to 1, we get that $-0.5 \leq a + b \leq 0.5$ and $-\sqrt{3}/2 \leq a - b \leq \sqrt{3}/2$. This is a rectangular region in the (a, b)-plane.

(i) Assume $a = 0.487654123$. Then we get:

```
>> A=[0.5 b 0 a; a 0.5 b 0; 0 a 0.5 b; b 0 a 0.5]

A =

    0.5000    0.0123         0    0.4877
    0.4877    0.5000    0.0123         0
         0    0.4877    0.5000    0.0123
    0.0123         0    0.4877    0.5000

>> eig(A)

ans =

        0
   1.0000
   0.5000 + 0.4753i
   0.5000 - 0.4753i

>> det(A)

ans =

     0
```

This coresponds to (2d) and (2f).

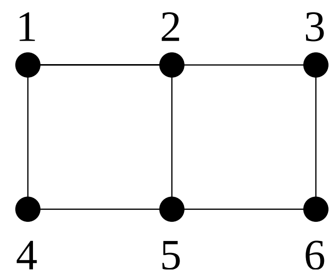
(j)
```
>> A^100

ans =

    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
```

It tends to $(1/4, 1/4, 1/4, 1/4)$; this is the eigenvector corresponding to $\lambda = 1$ ($\omega = 1$) properly scaled so that the sum of the entries equal to 1.

3. (9 pts) Consider the 2×3 grid shown below. Assume a mouse starts at vertex 1. At every step, the mouse either stays where it is with probability 0.5 or moves to an adjacent vertex selected uniformly among the current neighbors.



(a) The transition matrix A for this Markov Chain is:

$$\begin{bmatrix} 1/2 & 1/6 & 0 & 1/4 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 1/6 & 0 \\ 0 & 1/6 & 1/2 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/2 & 1/6 & 0 \\ 0 & 1/6 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/4 & 0 & 1/6 & 1/2 \end{bmatrix}$$

(b) The sum of the eigenvalues of A is equal to the trace, which is 3.

(c) `A =`

```
    0.5000    0.1667         0    0.2500         0         0
    0.2500    0.5000    0.2500         0    0.1667         0
         0    0.1667    0.5000         0         0    0.2500
    0.2500         0         0    0.5000    0.1667         0
         0    0.1667         0    0.2500    0.5000    0.2500
         0         0    0.2500         0    0.1667    0.5000

>> eig(A)

ans =

    1.0000
   -0.0000
    0.7500
    0.2500
    0.5833
    0.4167
```

If we were doing exact arithmetic, we would get $1, 0, 3/4, 1/4, 7/12, 5/12$.

(d) The steady state probabilities will be given by the eigenvector corrresponding to $\lambda = 1$ appropriately scaled:

```
>> [L,V]=eig(A)

L =

    0.3430    0.3430   -0.5000   -0.5000   -0.2887    0.2887
    0.5145   -0.5145    0.0000    0.0000   -0.5774   -0.5774
    0.3430    0.3430    0.5000    0.5000   -0.2887    0.2887
    0.3430   -0.3430   -0.5000    0.5000    0.2887    0.2887
    0.5145    0.5145   -0.0000    0.0000    0.5774   -0.5774
    0.3430   -0.3430    0.5000   -0.5000    0.2887    0.2887


V =

    1.0000         0         0         0         0         0
         0   -0.0000         0         0         0         0
         0         0    0.7500         0         0         0
         0         0         0    0.2500         0         0
         0         0         0         0    0.5833         0
         0         0         0         0         0    0.4167

>> L(:,1)/sum(L(:,1))

ans =

    0.1429
    0.2143
    0.1429
    0.1429
    0.2143
    0.1429
```

This is the vector $(1/7, 3/14, 1/7, 1/7, 3/14, 1/7)$. Thus the steady-state probability that the mouse is on either of the middle vertices is $3/14 + 3/14 = 3/7 = 0.428571\cdots$.
We verify that indeed the columns of A^k tend to the scaled eigenvector of A corresponding to $\lambda = 1$:

```
>> A^100

ans =

    0.1429    0.1429    0.1429    0.1429    0.1429    0.1429
    0.2143    0.2143    0.2143    0.2143    0.2143    0.2143
    0.1429    0.1429    0.1429    0.1429    0.1429    0.1429
    0.1429    0.1429    0.1429    0.1429    0.1429    0.1429
    0.2143    0.2143    0.2143    0.2143    0.2143    0.2143
```

```
0.1429    0.1429    0.1429    0.1429    0.1429    0.1429
```

4. (4 pts.) First, let us compute the eigenvalues of K. The characteristic polynomial is $-\lambda(i-\lambda)-(-1+i)(1+i)=\lambda^2-i\lambda+2$, and its roots are $\lambda_1=-i$ and $\lambda_2=2i$. The eigenvalues are pure imaginary. The eigenvectors are $v_1=(-1-i,1)$ and $v_2=(0.5+0.5i,1)$; thus the matrix of eigenvectors is:

$$V=\begin{bmatrix} -1-i & 0.5+0.5i \\ 1 & 1 \end{bmatrix}.$$

Thus we have that

$$K=V\begin{bmatrix} -i & 0 \\ 0 & 2i \end{bmatrix}V^{-1}.$$

We need V^H and not V^{-1}. But the columns of V (the eigenvectors) are orthogonal and we see that $V^HV=\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}$. Thus we just have to scale the eigenvectors (to norm 1) to get

$$U=\begin{bmatrix} \frac{1}{\sqrt{3}}(-1-i) & \frac{1}{\sqrt{3/2}}(0.5+0.5i) \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3/2}} \end{bmatrix}.$$

And now we have that

$$K=U\begin{bmatrix} -i & 0 \\ 0 & 2i \end{bmatrix}U^H.$$

There are actually other solutions for U as we can multiply U by any diagonal matrix with all diagonal elements of modulus 1 such as:

$$\begin{bmatrix} -i & 0 \\ 0 & \frac{\sqrt{2}}{2}(1-i) \end{bmatrix}.$$